Discrete Mathematics 310 (2010) 3072–3079

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Combinatorics of a class of groups with cyclic presentation

Agnese Ilaria Telloni

Dipartimento di Matematica, Università di Modena e Reggio E., Via Campi 213/B, 41100 Modena, Italy

ARTICLE INFO

Article history:
Received 9 September 2008
Received in revised form 28 February 2009
Accepted 10 March 2009
Available online 8 April 2009

Keywords:
3-manifolds
Cyclic branched coverings
Orbifolds
Fundamental groups
(1, 1)-knots
Torus knots

ABSTRACT

We study a family of combinatorial closed 3-manifolds obtained from polyhedral 3-balls, whose finitely many boundary faces are glued together in pairs. Then we determine geometric presentations of their fundamental groups, and find conditions under which such groups are infinite and/or aspherical. Moreover, we show that our presentations are a natural generalization of those considered by Prishchepov in [M.I. Prishchepov, Asphericity, atoricity and symmetrically presented groups, Comm. Algebra 23 (13) (1995) 5095–5117]. Finally we illustrate some geometric and topological properties of the constructed manifolds, as, for example, a combinatorial description of them as cyclic coverings of the 3-sphere branched over some specified classes of knots.

1. Introduction

Let F_n be the free group of rank n on generators $x_1, \ldots, x_n$. Let η denote the automorphism of order n such that $\eta(x_i) = x_{i+1}$, where the subscripts are taken modulo n. A group is said to have a *cyclic presentation* if it is isomorphic to

$$G_n(w) = \langle x_1, \ldots, x_n : w = 1, \eta(w) = 1, \ldots, \eta^{n-1}(w) = 1\rangle$$

for some positive integer n, where w is a reduced word in the alphabet $X = \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$ (see [9]). The automorphism η of F_n naturally induces an automorphism θ of order n on $G_n(w)$. Let $E_n(v)$ be the *split extension group* of $G_n(w)$ by the cyclic group $\mathbb{Z}_n$ generated by θ. Then $E_n(v)$ admits a finite presentation with two generators x and θ and two relations $\theta^n = 1$ and $v(x, \theta) = 1$. The second relation is obtained by setting $x_{i+1} = \theta^{-1}x_i\theta = \theta^{-i}x\theta^i$ and $x_0 = x$ in the word w. There is a short exact sequence $1 \longrightarrow G_n(w) \longrightarrow E_n(v) \longrightarrow \mathbb{Z}_n \longrightarrow 1$ which induces the isomorphism $E_n(v)/G_n(w) \cong Z_n$. In particular, $G_n(w)$ is finite if and only if $E_n(v)$ is finite. One of the motivations for the study of cyclically presented groups is their connection with the topology of closed connected orientable 3-manifolds (see, for example, [6] and its references).

Let us consider a family of cyclically presented groups depending on five nonnegative integers:

$$P(r, n, k, s, q) = \langle x_1, \ldots, x_n : x_i x_{i+q} \ldots x_{i+q(r-1)} = x_{i+k} x_{i+k+q} \ldots x_{i+k+q(s-1)} \quad (i = 1, \ldots, n)\rangle$$

where the indices are taken modulo n, $s \geq 1$, $r \geq 2$ and $1 \leq q < n$. This presentation coincides with $P(r, n, k + 1, s, q)$ as given by Prishchepov in [14]. He gave arithmetic conditions on the parameters which imply the asphericity and/or the atoricity of the presentations. For particular choices of parameters, these groups are known: $P(2, n, k, 1, m)$ are the *generalized Fibonacci groups* $G_n(m, k)$ introduced in [4], and studied in [1,5,19]; $P(r, n, 1, r-1, 2)$ are the *generalized Sieradski groups* $S(r, n)$ (see [16] for $r = 2$, and [3] for the general case); $P(r, n, r, 1, 1)$ are the *Fibonacci groups* $F(r, n)$ (see [10]); $P(r, n, r - 1 + k, 1, 1)$ and $P(r, n, r, k, 1)$ are two series of *groups of Fibonacci type* $F(r, n, k)$ and $H(r, n, k)$, respectively,

E-mail addresses: agnesetelloni@libero.it, agneseilaria.telloni@unimo.it.

0012-365X/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2009.03.011

which were considered and studied in [2,18]. In this paper we consider a natural generalization of the Prishchepov groups. Let $\widetilde{P}(r,n,k,s,p,q)$ denote the group with generators $x_1,\ldots,x_n$ and cyclically defined relations

$$x_i x_{i+q} \ldots x_{i+q(r-1)} = x_{i+k} x_{i+k+p} \ldots x_{i+k+p(s-1)}$$

for $i = 1,\ldots,n$, where the subscripts are taken modulo n, $s \geq 1$, $r \geq 2$ and $0 \leq p,q < n$. Throughout the paper, such groups will be called *generalized Prishchepov groups*. We determine certain arithmetic conditions on the parameters for which these groups correspond to spines of closed connected orientable 3-manifolds. Recall that a *spine* of a closed connected orientable 3-manifold M is a connected 2-polyhedron $X \subset M$ such that $M\backslash$(open 3-cell) collapses onto X. Our manifolds are constructed by using polyhedral 3-balls whose finitely many boundary faces are glued together in pairs. Then we study some topological and covering properties of the constructed manifolds. In particular, we show that our manifolds are cyclic branched coverings of lens spaces (possibly, $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{S}^3$), branched over some specified classes of knots. Furthermore, we determine the split extension group of $\widetilde{P}(r,n,k,s,p,q)$, and investigate under what conditions the generalized Prishchepov groups are infinite and/or aspherical. We answer such algebraic questions by considering the topological properties which arise from our combinatorial manifolds, and the connection between cyclically presented groups and the fundamental groups of some cyclic branched coverings.

2. The combinatorial manifolds $M_{n,p,q,k}$

Let us consider a triangulated polyhedron $P_{n,p,q}$, $n \geq 1$, $p,q \geq 0$, which represents a cellular decomposition of a 3-ball. The boundary of $P_{n,p,q}$ consists of n faces F_i, $i = 1,\ldots,n$, in the northern hemisphere, and n faces F'_{i+k} in the southern hemisphere:

$$F_i : \ c_i^0 c_i^1 \ldots c_i^q a_{i+1} a_i^{-1} (b_i^p)^{-1} \ldots (b_i^1)^{-1} (b_i^0)^{-1}$$
$$F'_{i+k} : \ d_{i+k} c_{i+k}^0 \ldots c_{i+k}^q (b_{i+k+1}^p)^{-1} \ldots (b_{i+k+1}^0)^{-1} d_{i+k+1}^{-1}$$

where $0 \leq k < n$ and the suffices are taken modulo n. Each face has exactly $p+q+2$ oriented edges, as listed above; the sequence of the oriented edges in ∂F_i (resp. $\partial F'_{i+k}$) is determined by using the counterclockwise (resp. clockwise) orientation in F_i (resp. F'_{i+k}). For every i (mod n) the face F_i intersects F_{i+1} in the edge a_{i+1}, and F'_{i+k} intersects F'_{i+k+1} in d_{i+k+1}. Every face F_i intersects F'_{i-1} (resp. F'_i) in the sequence of edges $b_i^0,\ldots,b_i^p$ (resp. $c_i^0,\ldots,c_i^q$). The boundary faces of the polyhedron $P_{n,p,q}$ are identified in pairs by the affine transformations $x_i : F_i \longrightarrow F'_{i+k}$, for $i = 1,\ldots,n$, and some $0 \leq k < n$. The map x_i realizes the pairing after a shift of k polygons. The oriented edges of F_i are identified to those of F'_{i+k} via x_i in the order specified by the above sequences. Thus $c_i^0 \to d_{i+k}$, $c_i^1 \to c_{i+k}^0,\ldots,c_i^q \to c_{i+k}^{q-1}$, $a_{i+1} \to c_{i+k}^q$, $a_i^{-1} \to (b_{i+k+1}^p)^{-1}$, $(b_i^p)^{-1} \to (b_{i+k+1}^{p-1})^{-1},\ldots,(b_i^1)^{-1} \to (b_{i+k+1}^0)^{-1}$, and $(b_i^0)^{-1} \to d_{i+k+1}^{-1}$. Let $M_{n,p,q,k}$ denote the quotient space obtained from $P_{n,p,q}$ by the above-described pairing of its boundary faces. This space is not a closed 3-manifold in general; there could be some singular points arising from the vertices of the polyhedron $P_{n,p,q}$. By a theorem of Seifert and Threlfall (see [15], Chp. 9, Theorem III), the quotient space $M_{n,p,q,k}$ is a closed connected orientable 3-manifold if and only if its Euler characteristic vanishes. Using this result, we have the following

Theorem 2.1. *For every $n \geq 1$, $p,q \geq 0$ and $0 \leq k < n$, the quotient combinatorial space $M_{n,p,q,k}$ obtained from the polyhedron $P_{n,p,q}$ by pairwise identification of its boundary faces under the above-defined pairing is a closed connected orientable 3-manifold if and only if $(p-q)k + p + 3 \equiv 0 \pmod{n}$.*

Proof. The quotient space has a natural cellular decomposition arising from that of the polyhedron $P_{n,p,q}$ via the squashing map. Of course, it has n 2-cells and one 3-cell. Now we show that the cellular decomposition of $M_{n,p,q,k}$ has exactly one 0-cell (vertex). Let E_i and N – the north pole – (resp. S – the south pole – and A_i) be the first and the second end of the oriented edge a_i (resp. d_i) in $P_{n,p,q}$ (here the subscripts are always considered modulo n). By construction, A_i is also the first vertex of the edges b_i^0 and c_i^0, and E_i (resp. E_{i+1}) is the second vertex of b_i^p (resp. c_i^q). Denote by B_i^1 and C_i^1 the second vertices of b_i^0 and c_i^0, respectively, and by B_i^p and C_i^q the first vertices of b_i^p and c_i^q, respectively. Finally, let B_i^j and B_i^{j+1} (resp. C_i^ℓ and $C_i^{\ell+1}$) be the first and the second end of b_i^j (resp. c_i^ℓ) for $0 < j < p$ and $0 < \ell < q$ (in the case of positive integers p and q). The pairwise identification of the boundary faces of $P_{n,p,q}$ determines the following sequence of equivalent vertices: $A_i \to S$, $C_i^1 \to A_{i+k}$, $C_i^\ell \to C_{i+k}^{\ell-1}$, $E_{i+1} \to C_{i+k}^q$, $N \to E_{i+k+1}$, $E_i \to B_{i+k+1}^p$, $B_i^j \to B_{i+k+1}^{j-1}$, and $B_i^1 \to A_{i+k+1}$, $i = 1,\ldots,n$, $2 \leq \ell \leq q$, and $2 \leq j \leq p$.

For every $i = 1,\ldots,n$, $A_i \xrightarrow{x_i} S$ and $E_i \xrightarrow{x_{i-k-1}^{-1}} N$ imply that all vertices A_i (resp. E_i) are equivalent to S (resp. N) in the quotient space, written $A_i \equiv S$ and $E_i \equiv N$. Then we get

$$S \equiv A_{i+k} \xrightarrow{x_i^{-1}} C_i^1 \xrightarrow{x_{i-k}^{-1}} C_{i-k}^2 \xrightarrow{x_{i-2k}^{-1}} \cdots \xrightarrow{x_{i-(q-1)k}^{-1}} C_{i-(q-1)k}^q \xrightarrow{x_{i-qk}^{-1}} E_{i-qk+1} \equiv N,$$

hence all the vertices A_i, S, N and C_i^ℓ are equivalent, for $i = 1,\ldots,n$, and $0 < \ell \leq q$. Furthermore, we have

$$N \equiv E_i \xrightarrow{x_i} B_{i+k+1}^p \xrightarrow{x_{i+k+1}} B_{i+2(k+1)}^{p-1} \xrightarrow{x_{i+2(k+1)}} \cdots \xrightarrow{x_{i+(p-1)(k+1)}} B_{i+p(k+1)}^1 \xrightarrow{x_{i+p(k+1)}} A_{i+(p+1)(k+1)} \equiv N,$$

hence all the vertices B_i^j are equivalent to N for $i = 1, \ldots, n$, and $0 < j \le p$. This proves that there is only one vertex in the quotient cellular space $M_{n,p,q,k}$. On the other hand, the above-defined pairwise identification of the boundary faces yields the following sequence of equivalent edges:

$$a_i \xrightarrow{x_i} b^p_{i+k+1} \xrightarrow{x_{i+k+1}} \cdots \xrightarrow{x_{i+p(k+1)}} b^0_{i+(p+1)(k+1)} \xrightarrow{x_{i+(p+1)(k+1)}} d_{i+(p+2)(k+1)} \xrightarrow{x^{-1}_{i+(p+2)(k+1)-k}} c^0_{i+(p+2)(k+1)-k}$$
$$\xrightarrow{x^{-1}_{i+(p+2)(k+1)-2k}} \cdots \xrightarrow{x^{-1}_{i+(p+2)(k+1)-(q+1)k}} c^q_{i+(p+2)(k+1)-(q+1)k} \xrightarrow{x^{-1}_{i+(p+2)(k+1)-(q+2)k}} a_{i+(p+2)(k+1)-(q+1)k-k+1}.$$

This is a closed cycle for every $i = 1, \ldots, n$ if and only if $1 + (p+2)(k+1) - (q+2)k \equiv 0 \pmod n$. So the quotient space $M_{n,p,q,k}$ has exactly n 1-cells if and only if the arithmetic condition of the statement holds. In this case, the Euler characteristic of $M_{n,p,q,k}$ is zero, so it is a closed 3-manifold. □

Let $G_{n,p,q,k}$ be the fundamental group of the closed 3-manifold $M_{n,p,q,k}$, where the parameters are assumed to satisfy the condition in Theorem 2.1. Recall that a *spine* of a closed combinatorial 3-manifold M is a two-dimensional subpolyhedron such that $M\setminus$(open 3-cell) collapses onto it. It is well known that every closed 3-manifold admits spines with a single vertex, corresponding to suitable presentations of its fundamental group. By construction, the interior of the polyhedron $P_{n,p,q}$ becomes an open 3-ball whose boundary meets itself in the quotient manifold $M_{n,p,q,k}$ along an embedded 2-complex which is a spine of the manifold. Moreover, it is the canonical 2-complex corresponding to a special presentation of the fundamental group $G_{n,p,q,k}$ as described in the following results.

Theorem 2.2. *For every $n \ge 1$, $p, q \ge 0$, $0 \le k < n$ and $(p-q)k + p + 3 \equiv 0 \pmod n$, the fundamental group $G_{n,p,q,k}$ of the closed connected orientable 3-manifold $M_{n,p,q,k}$ admits a finite presentation with n generators $x_1, \ldots, x_n$ and n cyclically defined relations*

$$x_i x_{i+k+1} \ldots x_{i+(p+1)(k+1)} = x_{i+p+2+(p-q)k} x_{i+p+2+(p-q+1)k} \ldots x_{i+p+2+(p+1)k}$$

for $i = 1, \ldots, n$ (subscripts mod n). This presentation is geometric, that is, it corresponds to a spine of the manifold $M_{n,p,q,k}$.

Since the groups $G_{n,p,q,k}$ are just generalized Prishchepov groups for suitable parameters, we get

Corollary 2.3. *Under the arithmetic conditions of Theorem 2.2, the generalized Prishchepov group $\widetilde{P}(p+2, n, p+2+(p-q)k, q+2, k, k+1)$ is the fundamental group of a closed connected orientable 3-manifold.*

We remark that the group in Corollary 2.3. is isomorphic to the group $\widetilde{P}(q+2, n, 1, p+2, k+1, k)$. We recall that a *Seifert manifold* Σ is uniquely characterized by a system of invariants $(\epsilon\ g\ \epsilon'\ :\ b\ (\alpha_1, \beta_1)\ (\alpha_2, \beta_2) \ldots (\alpha_r, \beta_r))$, where g is the genus of the base orbifold S, $\epsilon = O$ and $\epsilon' = o$ if Σ and S are orientable, respectively, $b = -(e_0 + \sum_{i=1}^{r} \beta_i/\alpha_i) \in \mathbb{Q}$, where e_o is the rational Euler number of the bundle, and (α_i, β_i) are the Seifert invariants of the ith exceptional fiber. For the theory of Seifert manifolds, we refer to the monograph of Orlik [13].

Theorem 2.4. *Assume the arithmetic conditions of Theorem 2.2 with $n = 2$. Then either* (i) *q, k are odd and then $M_{n,p,q,k}$ is homeomorphic to the Seifert manifold defined by the invariants*

$$(O\ 0\ o : -1\ (p+3, 1)\ (p+3, 1)\ ((q+3)/2, (q+1)/2));$$

or (ii) *p is odd and k is even and then $M_{n,p,q,k}$ is the Seifert manifold*

$$(O\ 0\ o : -1\ (q+3, 1)\ (q+3, 1)((p+3)/2, (p+1)/2)).$$

Proof. For $n = 2$, we have the following cases:

- k, p, q odd or k, q odd and p even. Then the condition $(p-q)k + p + 3 \equiv 0 \pmod 2$ holds, and Theorem 2.2. gives

$$\begin{aligned} G_{2,p,q,k} &= \langle x_1, x_2 : x_1^{p+2} = x_2(x_1x_2)^{\frac{q+1}{2}}, x_2^{p+2} = x_1(x_2x_1)^{\frac{q+1}{2}} \rangle \\ &\cong \langle x_1, x_2 : x_1^{p+3} = (x_1x_2)^{\frac{q+3}{2}} = x_2^{p+3} \rangle. \end{aligned}$$

This presentation corresponds to a spine of the manifold $M_{2,p,q,k}$. Moreover, $M_{2,p,q,k}$ is the tetrahedron manifold denoted by $M(p+3, 1; p+3, 1; (q+3)/2)$ in [17], and the result follows from Theorem 3.1 of [17].

- k even and p, q odd or k, q even and p odd. Then $(p-q)k + p + 3 \equiv 0 \pmod 2$ holds, and Theorem 2.2. gives

$$\begin{aligned} G_{2,p,q,k} &= \langle x_1, x_2 : (x_1x_2)^{\frac{p+1}{2}} x_1 = x_2^{q+2}, (x_2x_1)^{\frac{p+1}{2}} x_2 = x_1^{q+2} \rangle \\ &\cong \langle x_1, x_2 : x_1^{q+3} = (x_1x_2)^{\frac{p+3}{2}} = x_2^{q+3} \rangle. \end{aligned}$$

This presentation corresponds to a spine of the manifold $M_{2,p,q,k}$. So $M_{2,p,q,k}$ is the tetrahedron manifold $M(q+3, 1; q+3, 1; (p+3)/2)$, and the result follows from Theorem 3.1 of [17].

- k, p odd and q even or k odd and p, q even.
- k, p even and q odd or k, p, q even.

The last two cases do not occur since the condition $(p-q)k + p + 3 \equiv 0 \pmod 2$ is not verified. □

3. Algebraic properties of the groups $G_{n,p,q,k}$

From the cyclic presentation of $G_{n,p,q,k}$ in Theorem 2.2, we see that this group has a cyclic automorphism $\theta : x_i \to x_{i+1}$ of order n. This automorphism corresponds to the symmetry of order n (also denoted by θ) of the polyhedron $P_{n,p,q}$ such that $\theta : F_i \to F_{i+1}, F'_i \to F'_{i+1}$ (subscripts mod n). Let us consider the split extension group $E_{n,p,q,k}$ of $G_{n,p,q,k}$ with respect to the cyclic group $\mathbb{Z}_n$ generated by θ. Then we have

Theorem 3.1. *For every $n \geq 2$, $p, q \geq 0$, $0 \leq k < n$ and $(p-q)k + p + 3 \equiv 0 \pmod n$, the group $E_{n,p,q,k}$ has the finite presentation:*

$$E_{n,p,q,k} = \langle y, \theta : \theta^n = 1, (y\theta^{-1})^{p+3} = y^{q+3} \rangle.$$

In particular, if $(p+3, q+3) = 1$, then $E_{n,p,q,k}$ is isomorphic to the fundamental group of the three-dimensional orbifold whose underlying space is the 3-sphere and whose singular set is the torus knot $T(p+3, q+3)$.

Proof. Substituting $x_0 = x$ and $x_i = \theta^{-i}x\theta^i$ in the relations of $G_{n,p,q,k}$, yields

$$(x\theta^{-(k+1)})^{p+1}x\theta^{(p+1)(k+1)} = \theta^{(q-p)k-p-2}(x\theta^{-k})^{q+1}x\theta^{(p+1)k+p+2}$$

which is equivalent to the relation $(x\theta^{-(k+1)})^{p+1}x = \theta(x\theta^{-k})^{q+1}x\theta$ since $(q-p)k - p - 2 \equiv 1 \pmod n$ and $\theta^n = 1$. Setting $y = x\theta^{-k}$, with inverse relation $x = y\theta^k$, we obtain for $E_{n,p,q,k}$ a finite presentation with generators y and θ and the relation $(y\theta^{-1})^{p+1}y = \theta y^{q+2}\theta$, which is equivalent to that of the statement. Setting $a = y\theta^{-1}$ and $b = y$, which inverse relation $\theta = a^{-1}b$, we get $E_{n,p,q,k} = \langle a, b : (a^{-1}b)^n = 1, a^{p+3} = b^{q+3} \rangle$. Now, if $(p+3, q+3) = 1$, then $\langle a, b : a^{p+3} = b^{q+3} \rangle$ uniquely represents the group of the torus knot $T(p+3, q+3)$ in the 3-sphere, and $a^{-1}b$ represents a meridian around the knot. The second part of the statement follows. □

Theorem 3.2. *Assume $(p-q)k + p + 3 \equiv 0 \pmod n$ and $0 \leq k < n$. Then the group $G_{n,p,q,k}$ is infinite if and only if either $n \geq 3$ or $n = 2$ and $(p, q) \notin \{(0, 1), (1, 0)\}$. Otherwise, it is isomorphic to the finite group $SL(2, 3)$ of order 24.*

Proof. The group $G_{n,p,q,k}$ is infinite if and only if $E_{n,p,q,k} = \langle a, b : (a^{-1}b)^n = 1, a^{p+3} = b^{q+3} \rangle$ is infinite. The group $E_{n,p,q,k}$ covers the triangle group of type $(n, p+3, q+3)$ defined by $\langle a, b : a^{p+3} = b^{q+3} = (ab)^n = 1 \rangle$. It is known that the last group is infinite if and only if $1/n + 1/(p+3) + 1/(q+3) \leq 1$ (see [8], Section 6.4). This shows that $G_{n,p,q,k}$ is infinite unless possibly when $1/n + 1/(p+3) + 1/(q+3) > 1$. So the open cases are $G_{2,0,1,1}$ and $G_{2,1,0,0}$. In each case we show that these define the group $SL(2, 3)$. By Theorem 2.2 we obtain $G_{2,0,1,1} \cong \langle x_1, x_2 : x_1^2 = x_2x_1x_2, x_2^2 = x_1x_2x_1 \rangle$ and $G_{2,1,0,0} \cong \langle x_1, x_2 : x_1x_2x_1 = x_2^2, x_2x_1x_2 = x_1^2 \rangle$, so $G_{2,0,1,1} \cong G_{2,1,0,0} \cong \langle x_1, x_2 : x_1^3 = (x_1x_2)^2 = x_2^3 \rangle$. But this is precisely the *binary polyhedral group* $\langle \ell, m, n \rangle = \langle R, S, T : R^\ell = S^m = T^n = RST \rangle$, where $R = x_1x_2$, $S = x_1$, $T = x_2$, $m = n = 3$, and $\ell = 2$ (see [8], Section 6.5, formula (6.51), p. 68). Now it is known that $\langle 2, 3, 3 \rangle$ is isomorphic to the special linear group $SL(2, 3)$ by [8], Section 7.7, formula (7.73), p. 98. □

Corollary 3.3. *Under the arithmetic conditions of Theorem 3.2, the generalized Prishchepov group $\widetilde{P}(p+2, n, p+2+(p-q)k, q+2, k, k+1)$ is infinite if and only if either $n \geq 3$ or $n = 2$ and $(p, q) \notin \{(0, 1), (1, 0)\}$.*

4. Covering properties of the manifolds $M_{n,p,q,k}$

A knot K in a lens space $L(\xi, \eta)$ is said to be a $(1, 1)$-*knot* if there exists a genus one Heegaard splitting $(L(\xi, \eta), K) = (V_1, K_1) \cup (V_2, K_2)$, where V_i is a solid torus and $K_i \subset V_i$ is a properly embedded trivial arc, for $i = 1, 2$, and $\phi : (\partial V_2, \partial K_2) \to (\partial V_1, \partial K_1)$ is an attaching homeomorphism. An arc K properly embedded in a solid torus V is said to be *trivial* if there is a disk D in V with $K \subset \partial D$ and $\partial D \setminus K \subset \partial V$ (see, for example, [7]). Set $W_i = (V_i, K_i)$, $i = 1, 2$. We call the pair (W_1, W_2) a $(1, 1)$-*splitting* of $(L(\xi, \eta), K)$.

Theorem 4.1. *Assume $n \geq 1$, $p, q \geq 0$, $0 \leq k < n$ and $(p-q)k + p + 3 \equiv 0 \pmod n$. Then the closed 3-manifold $M_{n,p,q,k}$ is the n-fold cyclic covering of the lens space $L(|p-q|, 1)$ (in particular, $\mathbb{S}^1 \times \mathbb{S}^2$ for $p = q$, and $\mathbb{S}^3$ for $|p-q| = 1$), branched over a $(1, 1)$-knot $K_{p,q}$ only depending on the integers p and q.*

Proof. Let us consider the above automorphism θ of $G_{n,p,q,k}$, and denote the corresponding homeomorphism of $M_{n,p,q,k}$ also by θ. Since θ corresponds to the rotational symmetry of the polyhedron $P_{n,p,q}$, we see that the $\frac{1}{n}$-piece $\Pi_{p,q}$ of $P_{n,p,q}$, pictured in Fig. 1(a), is the fundamental polyhedron for the quotient space $M_{n,p,q,k}/\langle\theta\rangle$. This orbifold has the manifold $M_{1,p,q,0}$ as underlying topological space. The singular set of the orbifold is a knot $K_{p,q}$ arising from the north–south axis of the polyhedron $P_{n,p,q}$. Moreover, the fundamental group of the orbifold is isomorphic to the split extension group $E_{n,p,q,k}$ obtained in Theorem 3.1. After the identification of the sequence of edges $S\,A_1\,B_1^1 \dots B_1^p\,E_1\,N$ with the sequence $S\,A_2\,B_2^1 \dots B_2^p\,E_2\,N$ in this order, we can redraw the faces F and F' as the regions on the 2-sphere. In this case, we get a Heegaard diagram of $M_{1,p,q,0}$,

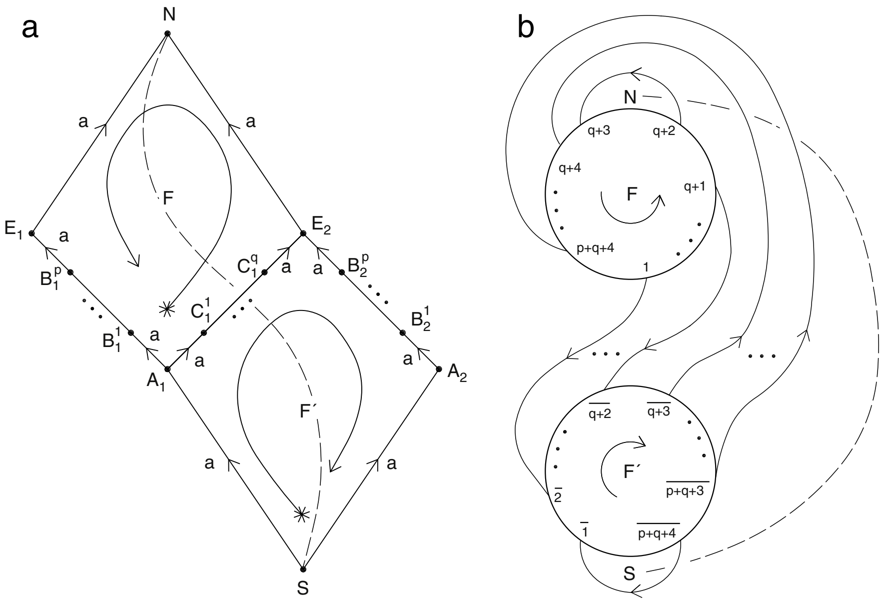

Fig. 1. The $\frac{1}{n}$-piece $\Pi_{p,q}$ of $P_{n,p,q}$ and a Heegaard diagram of the underlying closed manifold $M_{1,p,q,0}$; the singular set $K_{p,q}$ arises from the dotted arc NS.

drawn in Fig. 1(b). The axis of the rotation θ is pictured as a dotted curve in $\Pi_{p,q}$; it lies inside the ball whose boundary is being identified as indicated in Fig. 1(a).

The manifold $M_{1,p,q,0}$ has Heegaard genus ≤ 1, and its fundamental group has the presentation $G_{1,p,q,0} = \langle a : a^{p+2} = a^{q+2}\rangle \cong \mathbb{Z}_{|p-q|}$. From the Heegaard diagram in Fig. 1(b), we see that $M_{1,p,q,0}$ is the lens space $L(|p-q|, 1)$. Let V_1 be a regular neighborhood in $M_{1,p,q,0}$ of the 1-skeleton of $M_{1,p,q,0}$, and V_2 the closure of the complement of V_1 in $M_{1,p,q,0}$. Then the singular set $K_{p,q}$ (which is the image of the north–south axis of $\Pi_{p,q}$ in the quotient orbifold $M_{n,p,q,k}/\langle\theta\rangle$) decomposes into the union of two trivial arcs K_1 and K_2 such that $K_i \subset V_i$, $i = 1, 2$. Then the pair (W_1, W_2), where $W_i = (V_i, K_i)$, gives a $(1, 1)$-splitting of $(L(|p-q|, 1), K_{p,q})$. Hence the singular set $K_{p,q}$ is a $(1, 1)$-knot. □

Theorem 4.2. *For $n \geq 1$, $p, q \geq 0$, $p = q \pm 1$, $0 \leq k < n$ and $(p-q)k + p + 3 \equiv 0 \pmod n$, the closed connected orientable 3-manifold $M_{n,p,q,k}$ is the n-fold cyclic covering of the 3-sphere branched over the torus knot $T(q+3\pm 1, q+3)$.*

Proof. By Theorem 4.1 $M_{n,p,q,k}$ is the n-fold cyclic covering of the 3-sphere branched over a $(1, 1)$-knot $K_{p,q}$. In other words, $M_{n,p,q,k}$ is the n-fold cyclic covering of the orbifold $\mathcal{O}(K_{p,q}, n)$ whose underlying space is $\mathbb{S}^3$ and whose singular set is $K_{p,q}$ with branching index n. The split extension group $E_{n,p,q,k}$ is the fundamental group of the orbifold $\mathcal{O}(K_{p,q}, n)$. By Theorem 3.1 it is isomorphic to the fundamental group of the closed orbifold $\mathcal{O}(T(q+3\pm 1, q+3), n)$ whose underlying space is $\mathbb{S}^3$ and whose singular set is the torus knot $T(q+3\pm 1, q+3)$ with branching index n. Note that we can always choose the path represented by $a^{-1}b$ (see the proof of Theorem 3.1) as a meridian of the torus knot $T(q+3\pm 1, q+3)$ since $\det\begin{pmatrix} -1 & 1 \\ -(q+3\pm 1) & q+3 \end{pmatrix} = -(q+3) + q + 3 \pm 1 = \pm 1$. Now we prove that $\mathcal{O}(K_{p,q}, n)$ is homeomorphic to $\mathcal{O}(T(q+3\pm 1, q+3), n)$. In fact, a $(1, 1)$-knot is a two-generator knot (see, for example, [7]). Therefore a $(1, 1)$-knot in $\mathbb{S}^3$ is prime by [12]. Since prime knots in the 3-sphere are completely classified by their groups (see for example [11], Theorem 6.1.12, p. 76), the proof is complete. □

Since the orbifolds $\mathcal{O}(T(q+3\pm 1, q+3), n)$ are fibered, the manifolds in Theorem 4.2 are fibered, too. Under the conditions of Theorem 3.2 such manifolds have infinite fundamental groups, so they are aspherical by [13], Proposition 3, p. 93. Let $\varphi = \langle a_1, \ldots, a_n : r_1 = 1, \ldots, r_m = 1\rangle$ be a finite group presentation. There is a canonical 2-complex K_φ associated with φ. Its 1-skeleton is a bouquet of oriented circles, one for each a_i. The 2-cells of K_φ correspond bijectively to the relators r_j which determine closed curves as the corresponding attaching maps. We say that a finite group presentation φ is *(topologically) aspherical* if the canonical complex K_φ is aspherical, that is, $\pi_i(K_\varphi) \cong 0$ for every $i \geq 2$.

Corollary 4.3. *Under the conditions of Corollary 3.3 and Theorem 4.2, the generalized Prishchepov group presentation $\widetilde{P}(p+2, n, p+2+(p-q)k, q+2, k, k+1)$ is aspherical.*

Example. Let us consider the polyhedron $P_{6,3,2}$ with side-pairing of its boundary faces described in Fig. 2(a). The quotient space $M_{6,3,2,0}$ is a closed connected orientable 3-manifold. The rotational symmetry of the polyhedron $P_{6,3,2}$ around the North–South axis shows that $M_{6,3,2,0}$ is the 6-fold cyclic covering of the 3-sphere branched over a specified $(1, 1)$-knot $K_{3,2}$.

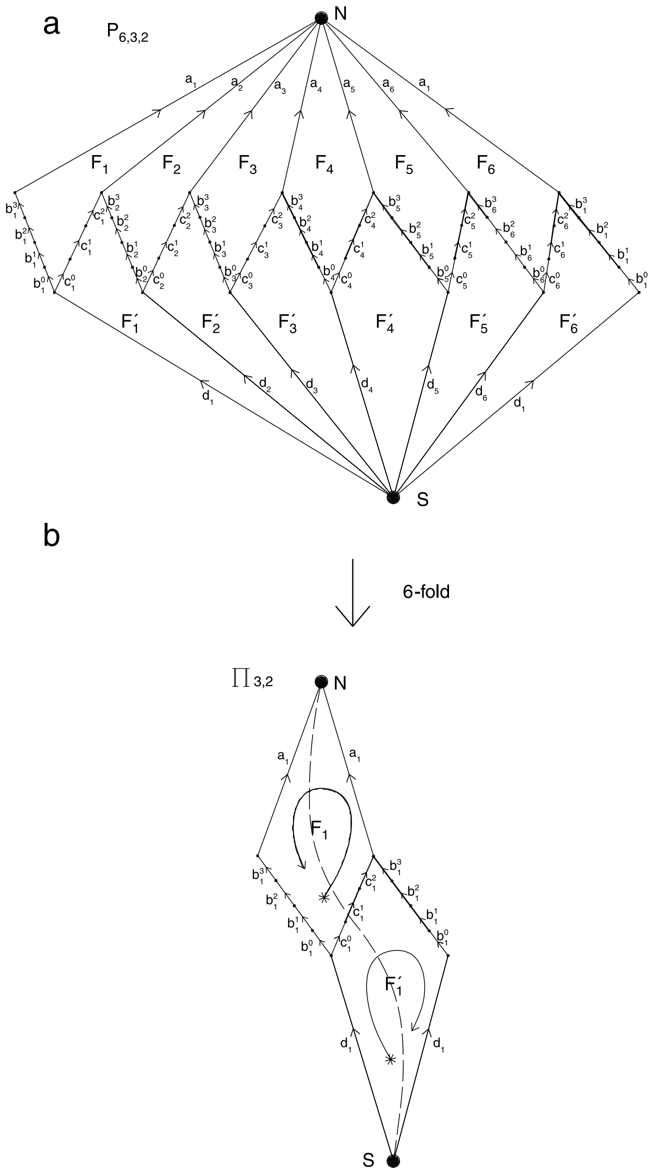

Fig. 2. The polyhedral scheme $P_{6,3,2}$, the 1/6-piece $\Pi_{3,2}$ and the orbifold $\mathcal{O}(K_{3,2}, 6)$.

The fundamental group of $M_{6,3,2,0}$ has a presentation with generators $x_1, x_2, \ldots, x_6$ and cyclic relations $x_i x_{i+1} x_{i+2} x_{i+3} x_{i+4} = x_{i+5}^4$ for every $i = 1, 2, \ldots, 6$ (subscripts mod 6). The abelianized group of $\pi_1(M_{6,3,2,0})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_{434}$. The orbifold $\mathcal{O}(K_{3,2}, 6)$ with underlying space $\mathbb{S}^3$ and with singular set $K_{3,2}$ is depicted in Fig. 2(b). Here there is the 1/6-piece $\Pi_{3,2}$ of $P_{6,3,2}$. The singular set $K_{3,2}$ arises from the dotted arc NS.

A Heegaard diagram of the underlying closed 3-manifold $M_{1,3,2,0} \cong \mathbb{S}^3$ is drawn in Fig. 3. The dotted arc represents the singular set $K_{3,2}$; it lies inside the 3-ball on which boundary the Heegaard diagram has been flattened. The diagram is reducible by techniques of cancellation of handles. Fig. 4 is obtained from Fig. 3 by cancellation of the handle represented by the arc with ends 1 and $\bar{1}$, and the boundary of the disc D. Fig. 5 represents the singular set $K_{3,2}$ of the cyclic covering. By Reidemeister moves it is easy to see that $K_{3,2}$ is equivalent to the torus knot $T(6, 5)$.

Acknowledgements

I should like to thank the referees for their helpful comments and suggestions to improve the paper.

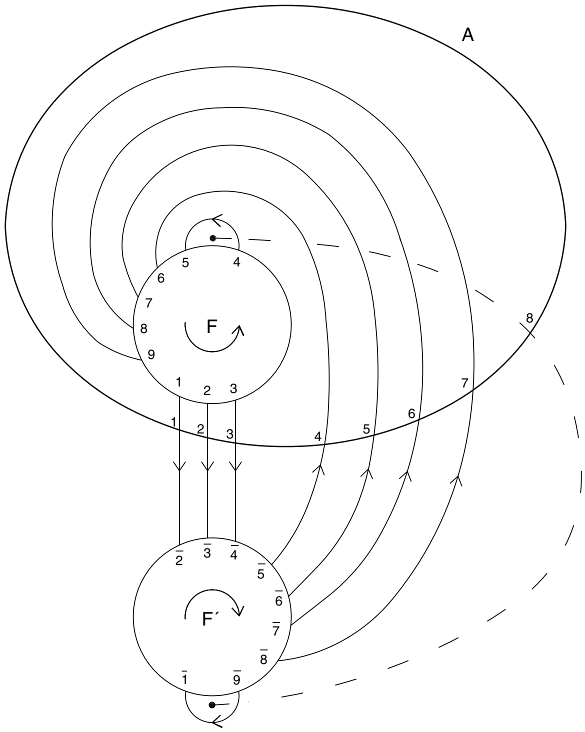

Fig. 3. A Heegaard diagram of $M_{1,3,2,0} \cong \mathbb{S}^3$.

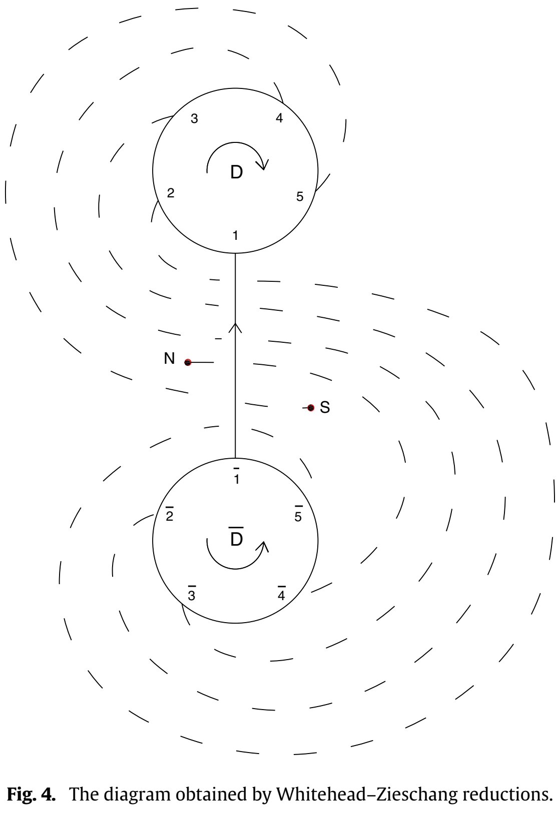

Fig. 4. The diagram obtained by Whitehead–Zieschang reductions.

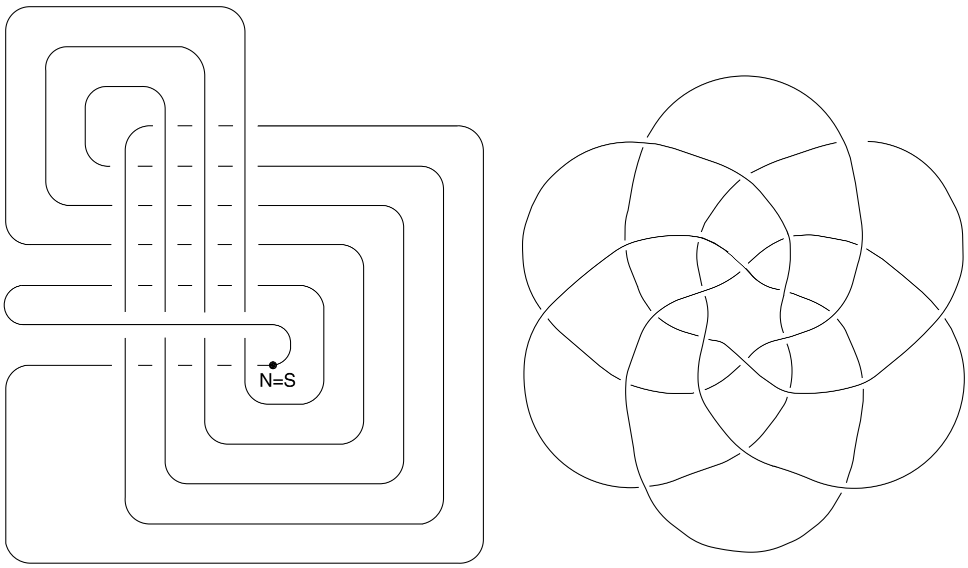

Fig. 5. The singular set $K_{3,2} = T(6, 5)$.

References

[1] V.G. Bardakov, A.Yu. Vesnin, A generalization of Fibonacci groups, Algebra Logic 42 (2) (2003) 73–91.
[2] C.M. Campbell, E.F. Robertson, A class of finitely presented groups of Fibonacci type, J. London Math. Soc. 11 (1975) 249–255.
[3] A. Cavicchioli, F. Hegenbarth, A.C. Kim, A geometric study of Sieradski groups, Algebra Colloq. 5 (2) (1998) 203–217.
[4] A. Cavicchioli, F. Hegenbarth, D. Repovš, On manifold spines and cyclic presentations of groups, in: Knot Theory (Warsaw), vol. 42, Banach Center Publ., 1998, pp. 49–55.
[5] A. Cavicchioli, E.A. O'Brien, F. Spaggiari, On some questions about a family of cyclically presented groups, J. Algebra 320 (2008) 4063–4072.
[6] A. Cavicchioli, D. Repovš, F. Spaggiari, Topological properties of cyclically presented groups, J. Knot Theory Ramifications 12 (2) (2003) 248–268.
[7] D.H. Choi, K.H. Ko, Parametrizations of 1-bridge torus knots, J. Knot Theory Ramifications 12 (4) (2003) 463–491.
[8] H.S.M. Coxeter, W.O.J. Moser, Generators and Relations for Discrete Groups, Springer-Verlag, Berlin, 1980.
[9] D.L. Johnson, Topics in the Theory of Group Presentations, in: London Math. Soc. Lect. Note Ser., vol. 42, Cambridge Univ. Press, Cambridge, 1980.
[10] D.L. Johnson, J.W. Wamsley, D. Wright, The Fibonacci groups, Proc. London Math. Soc. 29 (1974) 577–592.
[11] A. Kawauchi, A Survey of Knot Theory, Birkhäuser Verlag, Basel, Boston, Berlin, 1996.
[12] F. Norwood, Every two-generator knot is prime, Proc. Amer. Math. Soc. 86 (1982) 143–147.
[13] P. Orlik, Seifert Manifolds, Springer-Verlag, Berlin, Heidelberg, New York, 1972.
[14] M.I. Prishchepov, Asphericity, atoricity and symmetrically presented groups, Comm. Algebra 23 (13) (1995) 5095–5117.
[15] H. Seifert, W. Threlfall, A Text-book of Topology. Topology of 3-Dimensional Fibered Spaces, Academic Press, London, New York, 1980.
[16] A.J. Sieradski, Combinatorial squashings, 3-manifolds, and the third homology of groups, Invent. Math. 84 (1986) 121–139.
[17] F. Spaggiari, The combinatorics of some tetrahedron manifolds, Discrete Math. 300 (2005) 163–179.
[18] A. Szczepański, A. Vesnin, On generalized Fibonacci groups with an odd number of generators, Comm. Algebra 28 (2) (2000) 959–965.
[19] G. Williams, The aspherical Cavicchioli–Hegenbarth–Repovš generalized Fibonacci groups, J. Group Theory (2008) (in press).